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ACADEMIC PRESS

Journal of Approximation Theory 124 (2003) 96–108

JOURNAL OF Approximation Theory

http://www.elsevier.com/locate/jat

Uniform convergence of Cesàro means of negative order of double Walsh–Fourier series

Ushangi Goginava*

Department of Mechanics and Mathematics, Tbilisi State University, Chavchavadze str. 1, Tbilisi 0128, Georgia

Received 24 December 2002; accepted in revised form 26 June 2003

Communicated by András Kroo

Abstract

In this paper we prove that if $f \in C_W([0,1]^2)$ and the function f is bounded partial p-variation for some $p \in [1, +\infty)$ then the double Walsh–Fourier series of the function f is uniformly $(C; -\alpha, -\beta)$ summable $(\alpha + \beta < 1/p, \alpha, \beta > 0)$ in the sense of Pringsheim. If $\alpha + \beta \geqslant 1/p$ then there exists a continuous function f_0 of bounded partial p-variation on $[0,1]^2$ such that the Cesàro $(C; -\alpha, -\beta)$ means $\sigma_{n,m}^{-\alpha,-\beta}(f_0; 0, 0)$ of the double Walsh–Fourier series of f_0 diverge over cubes.

MSC: 42B08; 42C10

Keywords: Double Walsh–Fourier series; Cesàro means; Uniform summability

1. Introduction

Let $r_0(x)$ be a function defined by

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0, 1/2), \\ -1, & \text{if } x \in [1/2, 1), \end{cases} \quad r_0(x+1) = r_0(x).$$

The Rademachere system is defined by

$$r_n(x) = r_0(2^n x), \quad n \geqslant 1 \quad \text{and } x \in [0, 1).$$

*Fax: +99-532-536-583.

E-mail address: z_goginava@hotmail.com.

0021-9045/$ - see front matter
doi:10.1016/S0021-9045(03)00134-5

Let $w_0, w_1, \dots$ represent the Walsh functions, i.e. $w_0(x) = 1$ and if $k = 2^{n_1} + \cdots + 2^{n_s}$ is a positive integer with $n_1 > n_2 > \cdots > n_s$ then

$$w_k(x) = r_{n_1}(x) \cdots r_{n_s}(x).$$

The idea of using products of Rademacher's functions to define the Walsh system originated from Paley [9].

The Walsh–Dirichlet kernel is defined by

$$D_n(x) = \sum_{k=0}^{n-1} w_k(x).$$

Recall that [1, Chapter 1]

$$D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in [0, 1/2^n), \\ 0, & \text{if } x \in [1/2^n, 1). \end{cases} \tag{1}$$

We consider the double system $\{w_n(x) \times w_m(y) : n, m = 0, 1, 2, \dots\}$ on the unit square $I^2 = [0,1) \times [0,1)$.

If $f \in L(I^2)$, then

$$\hat{f}(n,m) = \int_0^1 \int_0^1 f(x,y) w_n(x) w_m(y)\, dx\, dy$$

is the (n,m)th Fourier coefficient of f.

The Cesàro $(C; \alpha, \beta)$-means of double Walsh–Fourier series are defined as follows:

$$\sigma_{n,m}^{\alpha,\beta}(f; x, y) = \frac{1}{A_n^\alpha A_m^\beta} \sum_{i=0}^{n} \sum_{j=0}^{m} A_{n-i}^\alpha A_{m-j}^\beta \hat{f}(i,j) w_i(x) w_j(y),$$

where

$$A_0^\alpha = 1, \quad A_n^\alpha = \frac{(\alpha+1)\cdots(\alpha+n)}{n!}, \quad \alpha \neq -1, -2, \dots \,.$$

Let $C_W(I^2)$ be the set of all functions $f : I^2 \to R$ that are uniformly continuous from the dyadic topology of I^2 to the usual topology of R with the norm [10, pp. 9–11]

$$\|f\|_C = \sup_{x,y \in I^2} |f(x,y)|.$$

The dyadic partial moduli of continuity of a function $f \in C_W(I^2)$ are defined by

$$\dot{\omega}_1(f; \delta_1) = \sup\{\|f(x \oplus u, y) - f(x,y)\|_C \colon 0 \leqslant u < \delta_1\},$$

$$\overset{\bullet}{\omega}_2(f;\delta_2) = \sup\{\|f(x,y\oplus v) - f(x,y)\|_C:\ 0\leqslant v<\delta_2\},$$

where $\oplus$ denotes dyadic addition [1, Chapter 1].

A function $f: I^2 \to R$ is said to be of bounded variation in the sense of Hardy ($f\in HBV(I^2)$) [6] if there exists a constant K such that for any partition

$$\Delta_1:\ 0\leqslant x_0<x_1<x_2<\cdots<x_n\leqslant 1,$$

$$\Delta_2:\ 0\leqslant y_0<y_1<y_2<\cdots<y_m\leqslant 1,$$

we have

$$V_{1,2}(f) = \sup_{\Delta_1\times\Delta_2} \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} |f(x_i,y_j) - f(x_{i+1},y_j) - f(x_i,y_{j+1}) + f(x_{i+1},y_{j+1})| \leqslant K,$$

$$V_1(f) = \sup_y \sup_{\Delta_1} \sum_{i=0}^{n-1} |f(x_i,y) - f(x_{i+1},y)| \leqslant K,$$

$$V_2(f) = \sup_x \sup_{\Delta_2} \sum_{i=0}^{n-1}\sum_{j=0}^{m-1} |f(x,y_j) - f(x,y_{j+1})| \leqslant K.$$

Definition 1. We say that the function $f: I^2\to R$ is bounded partial p-variation ($f\in PBV_p(I^2)$) if

$$V_1(f)_p = \sup_y \sup_{\Delta_1} \sum_{i=0}^{n-1} |f(x_i,y) - f(x_{i+1},y)|^p,$$

$$V_2(f)_p = \sup_x \sup_{\Delta_2} \sum_{j=0}^{m-1} |f(x,y_j) - f(x,y_{j+1})|^p$$

are finite.

Given a function $f(x,y)$, periodic in both variables with period 1, for $0\leqslant j<2^m$ and $0\leqslant i<2^n$ and integers $m,n\geqslant 0$ we set

$$\Delta_j^m f(x,y)_1 = f(x\oplus 2j2^{-m-1},y) - f(x\oplus(2j+1)2^{-m-1},y),$$

$$\Delta_i^n f(x,y)_2 = f(x,y\oplus 2i2^{-n-1}) - f(x,y\oplus(2i+1)2^{-n-1}),$$

$$\begin{aligned}\Delta_{ji}^{mn} f(x,y) &= \Delta_i^n(\Delta_j^m f(x,y)_1)_2 = \Delta_j^m(\Delta_i^n f(x,y)_2)_1\\ &= f(x\oplus 2j2^{-m-1}, y\oplus 2i2^{-n-1}) - f(x\oplus(2j+1)2^{-m-1}, y\oplus 2i2^{-n-1})\\ &\quad - f(x\oplus 2j2^{-m-1}, y\oplus(2i+1)2^{-n-1})\\ &\quad + f(x\oplus(2j+1)2^{-m-1}, y\oplus(2i+1)2^{-n-1}).\end{aligned}$$

Denote

$$W_m^{(1)}(f;x,y) = \sum_{j=1}^{2^m-1} \frac{1}{j^{1-\alpha}} |\Delta_j^m f(x,y)_1|,$$

$$W_n^{(2)}(f;x,y) = \sum_{i=1}^{2^n-1} \frac{1}{i^{1-\beta}} |\Delta_i^n f(x,y)_2|,$$

$$W_{mn}(f;x,y) = \sum_{j=1}^{2^m-1} \sum_{i=1}^{2^n-1} \frac{1}{j^{1-\alpha}} \frac{1}{i^{1-\beta}} |\Delta_{ji}^{mn} f(x,y)|.$$

2. Formulation of the problems

Jordan [7] introduced a class of functions of bounded variation and, applying it to the theory of the Fourier series, he proved that if a continuous function has bounded variation, then its Fourier series converges uniformly. In 1906 Hardy [6] generalized the Jordan criterion to the double Fourier series and introduced for the function of two variables the notion of bounded variation. He proved that if the continuous function of two variables has bounded variation (in the sense of Hardy), then its Fourier series converges uniformly in the sense of Pringsheim.[1] The analogous result for double Walsh–Fourier series is verified by Moricz [8]. The author [2] has proved that in Hardy's theorem there is no need to requere the boundedness of $V_{1,2}(f)$; moreover, it is proved that if f is continuous function and has bounded p-partial variation ($f \in PBV_p$) for some $p \in [1, +\infty)$ then its double trigonometric Fourier series converges uniformly on $[0, 2\pi]^2$ in the sense of Pringsheim. The analogous result for double Walsh–Fourier series is established in [3].

In [4] the following theorems are proved:

Theorem A. *Let $f \in C_W(I^2) \cap PBV_1$ and $\alpha + \beta < 1$, $\alpha, \beta \in (0,1)$. Then the double Walsh–Fourier series of the function f is uniformly $(C; -\alpha, -\beta)$ summable in the sense of Pringsheim.*

Theorem B. *Let $\alpha + \beta = 1, \alpha, \beta > 0$. Then there exists a continuous function $f_0 \in PBV_1$ such that the Cesàro $(C; -\alpha, -\beta)$ means $\sigma_{n,m}^{-\alpha,-\beta}(f_0; 0, 0)$ of the double Walsh–Fourier series of f_0 diverge over cubes.*

On the basis of the above facts the following problems arise naturally:

Let $f \in PBV_p(I^2) \cap C_W(I^2)$, for some $p \in [1, +\infty)$. Find all values of $\alpha, \beta \in (0,1)$ for which the uniform convergence of Cesàro $(C; -\alpha, -\beta)$ means of double Walsh–Fourier series of the function f holds.

[1] A double series is said to converge in the sense of Prinsgeim if its partial rectangular sums converge.

The solution of this problem is given by Theorems 1 and 2.

3. Main results

The main results of this paper are presented in the following propositions.

Theorem 1. *Let $f \in C_W(I^2) \cap PBV_p$, for some $p \in [1, +\infty)$ and $\alpha + \beta < 1/p, \alpha, \beta > 0$. Then the double Walsh–Fourier series of the function f is uniformly $(C; -\alpha, -\beta)$ summable in the sense of Pringsheim.*

Theorem 2. *Let $\alpha + \beta \geqslant 1/p, \alpha, \beta > 0$. Then there exists a continuous function $f_0 \in PBV_p$ such that the Cesàro $(C; -\alpha, -\beta)$ means $\sigma_{n,m}^{-\alpha,-\beta}(f_0; 0, 0)$ of the double Walsh–Fourier series of f_0 diverge over cubes.*

Theorems 1 and 2 imply

Theorem 3. *Let $\alpha, \beta \in (0, 1)$ and $p \in [1, \infty)$. For all double Walsh–Fourier series of class $C_W(I^2) \cap PBV_p$ to be uniformly $(C; -\alpha, -\beta)$ summable it is necessary and sufficient that*

$$\alpha + \beta < 1/p.$$

4. Auxiliary results

We shall need

Lemma 1 (Tevzadze [11]). *Let*

$$K_n^\alpha(t) = \frac{1}{A_n^\alpha} \sum_{j=0}^{n} A_{n-j}^\alpha w_j(t).$$

Then there exists a positive integer s, such that

$$\int_{\frac{2^{i-1}}{2^m}}^{\frac{2^i}{2^m}} |K_{2^m}^{-\alpha}(t)|\, dt \geqslant c(\alpha) 2^{i\alpha}$$

is fulfilled if $i < m - s$ for great m.

Lemma 2. *Let f be continuous 1-periodic functions on $[0, 1]$ and $\alpha \in (0, 1)$. Then*

$$\|\sigma_n^{-\alpha}(f) - f\|_C \leqslant c(\alpha) \omega\left(\frac{1}{n}, f\right) n^\alpha,$$

where $\omega(\delta, f)$ is the modulus of continuity.

The proof can be found in [5].

Lemma 3. *Let $f \in C_W(I^2)$ and*

$$W_n^{(1)}(f;x,y),\, W_m^{(2)}(f;x,y),\, W_{nm}(f;x,y) \to 0$$

uniformly with respect to x, y as $m, n \to \infty$. Then the double Walsh–Fourier series of the function f is uniformly $(C; -\alpha, -\beta)$ summable in the sense of Pringsheim.

The proof can be found in [3].

5. Proofs of main results

Proof of Theorem 1. To prove the theorem on the basis of Lemma 3 it suffices to show that

$$W_n^{(1)}(f;x,y),\, W_m^{(2)}(f;x,y),\, W_{nm}(f;x,y) \to 0$$

uniformly with respect to x, y as $m, n \to \infty$.

Using Abel Transformation we obtain

$$\begin{aligned} W_n^{(1)}(f;x,y) &= \sum_{i=1}^{2^n-1} \frac{1}{i^{1-\alpha}} |\Delta_i^n f(x,y)_1| \\ &= \sum_{i=1}^{2^n-2} \left(\frac{1}{i^{1-\alpha}} - \frac{1}{(i+1)^{1-\alpha}} \right) \sum_{j=1}^{i} |\Delta_j^n f(x,y)_1| \\ &\quad + \frac{1}{(2^n-1)^{1-\alpha}} \sum_{i=1}^{2^n-1} |\Delta_i^n f(x,y)_1| = \mathrm{I} + \mathrm{II}. \end{aligned} \tag{2}$$

Using Holder inequality, from the condition of the theorem we get

$$\begin{aligned} \mathrm{II} &\leqslant \frac{1}{(2^n-1)^{1/p-\alpha}} \left(\sum_{i=1}^{2^n-1} |\Delta_i^n f(x,y)_1|^p \right)^{1/p} \\ &= O\left(\frac{1}{(2^n-1)^{1/p-\alpha}} \right) = o(1) \quad \text{as } n \to \infty, \end{aligned} \tag{3}$$

$$\begin{aligned} \mathrm{I} &\leqslant c(\alpha) \sum_{i=1}^{2^n-2} \frac{1}{i^{2-\alpha}} \sum_{j=1}^{i} |\Delta_j^n f(x,y)_1| \\ &\leqslant c(\alpha) \left\{ \sum_{i=1}^{i(n)} \frac{1}{i^{2-\alpha}} \sum_{j=1}^{i} |\Delta_j^n f(x,y)_1| + \sum_{i=i(n)+1}^{2^n-2} \frac{1}{i^{1+1/p-\alpha}} \left(\sum_{j=1}^{i} |\Delta_j^n f(x,y)_1|^p \right)^{1/p} \right\} \\ &\leqslant c(\alpha) \left\{ \dot{\omega}_1\left(f, \frac{1}{2^n}\right) (i(n))^{\alpha} + \left(\frac{1}{i(n)} \right)^{1/p-\alpha} \right\} = o(1) \quad \text{as} \quad n \to \infty, \end{aligned} \tag{4}$$

where

$$\min_{1\leqslant m<2^n}\left\{\overset{\bullet}{\omega}_1\left(f,\frac{1}{2^n}\right)m^{\alpha}+\left(\frac{1}{m}\right)^{1/p-\alpha}\right\}$$
$$=\left\{\overset{\bullet}{\omega}_1\left(f,\frac{1}{2^n}\right)(i(n))^{\alpha}+\left(\frac{1}{i(n)}\right)^{1/p-\alpha}\right\}.$$

From (2)–(4) we obtain

$$W_n^{(1)}(f;x,y)\to 0 \tag{5}$$

uniformly with respect to x, y as $n\to\infty$.

Analogously we obtain

$$W_m^{(2)}(f;x,y)\to 0 \tag{6}$$

uniformly with respect to x, y as $m\to\infty$.

Using Hardy transformation, we obtain

$$\begin{aligned}
W_{mn}(f;x,y) &= \sum_{i=1}^{2^n-2}\sum_{j=1}^{2^m-2}\left(\frac{1}{i^{1-\alpha}}-\frac{1}{(i+1)^{1-\alpha}}\right)\left(\frac{1}{j^{1-\beta}}-\frac{1}{(j+1)^{1-\beta}}\right)\\
&\quad\times\sum_{l=1}^{i}\sum_{s=1}^{j}|\Delta_{ls}^{nm}f(x,y)|\\
&\quad+\left(\frac{1}{2^n-1}\right)^{1-\alpha}\sum_{j=1}^{2^m-2}\left(\frac{1}{j^{1-\beta}}-\frac{1}{(j+1)^{1-\beta}}\right)\sum_{l=1}^{2^n-1}\sum_{s=1}^{j}|\Delta_{ls}^{nm}f(x,y)|\\
&\quad+\left(\frac{1}{2^m-1}\right)^{1-\beta}\sum_{i=1}^{2^n-2}\left(\frac{1}{i^{1-\alpha}}-\frac{1}{(i+1)^{1-\alpha}}\right)\sum_{l=1}^{i}\sum_{s=1}^{2^m-1}|\Delta_{ls}^{nm}f(x,y)|\\
&\quad+\left(\frac{1}{2^n-1}\right)^{1-\alpha}\left(\frac{1}{2^m-1}\right)^{1-\beta}\sum_{i=1}^{2^n-1}\sum_{j=1}^{2^m-1}|\Delta_{ij}^{nm}f(x,y)|\\
&= \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}.
\end{aligned} \tag{7}$$

It is evident that

$$\sum_{l=1}^{i}\sum_{s=1}^{j}|\Delta_{ls}^{nm}f(x,y)|\leqslant 2j\sup_{y}\sum_{l=1}^{i}|\Delta_l^nf(x,y)_1| \tag{8}$$

and

$$\sum_{l=1}^{i}\sum_{s=1}^{j}|\Delta_{ls}^{nm}f(x,y)|\leqslant 2i\sup_{x}\sum_{s=1}^{j}|\Delta_s^mf(x,y)_2|. \tag{9}$$

Consequently,

$$\sum_{l=1}^{i}\sum_{s=1}^{j}|\Delta_{ls}^{nm}f(x,y)|$$
$$=\left(\sum_{l=1}^{i}\sum_{s=1}^{j}|\Delta_{ls}^{nm}f(x,y)|\right)^{\frac{\alpha}{\alpha+\beta}}\left(\sum_{l=1}^{i}\sum_{s=1}^{j}|\Delta_{ls}^{nm}f(x,y)|\right)^{\frac{\beta}{\alpha+\beta}}$$
$$\leqslant 2j^{\frac{\alpha}{\alpha+\beta}}\sup_{y}\left(\sum_{l=1}^{i}|\Delta_{l}^{n}f(x,y)_1|\right)^{\frac{\alpha}{\alpha+\beta}}$$
$$\times\, i^{\frac{\beta}{\alpha+\beta}}\left(\sup_{x}\sum_{s=1}^{j}|\Delta_{s}^{m}f(x,y)_2|\right)^{\frac{\beta}{\alpha+\beta}}. \tag{10}$$

Using Holder inequality, by (10) and from the condition of the theorem we get

$$\mathrm{IV}\leqslant 2\left(\frac{1}{2^n-1}\right)^{\frac{\alpha}{\alpha+\beta}-\alpha}\left(\frac{1}{2^m-1}\right)^{\frac{\beta}{\alpha+\beta}-\beta}$$
$$\times\sup_{y}\left(\sum_{l=1}^{2^n-1}|\Delta_{l}^{n}f(x,y)_1|\right)^{\frac{\alpha}{\alpha+\beta}}\left(\sup_{x}\sum_{s=1}^{2^m-1}|\Delta_{s}^{m}f(x,y)_2|\right)^{\frac{\beta}{\alpha+\beta}}$$
$$\leqslant 2\left(\frac{1}{2^n-1}\right)^{\frac{\alpha}{p(\alpha+\beta)}-\alpha}\left(\frac{1}{2^m-1}\right)^{\frac{\beta}{p(\alpha+\beta)}-\beta}$$
$$\times\sup_{y}\left(\sum_{l=1}^{2^n-1}|\Delta_{l}^{n}f(x,y)_1|^p\right)^{\frac{\alpha}{p(\alpha+\beta)}}\left(\sup_{x}\sum_{s=1}^{2^m-1}|\Delta_{s}^{m}f(x,y)_2|^p\right)^{\frac{\beta}{p(\alpha+\beta)}}$$
$$=O\left(\left(\frac{1}{2^n-1}\right)^{\frac{\alpha}{p(\alpha+\beta)}-\alpha}\left(\frac{1}{2^m-1}\right)^{\frac{\beta}{p(\alpha+\beta)}-\beta}\right)=o(1)\quad\text{as } n,m\to\infty. \tag{11}$$

$$\mathrm{III}\leqslant c(\alpha)\left(\frac{1}{2^m-1}\right)^{1-\beta}\sum_{i=1}^{2^n-2}\frac{1}{i^{2-\alpha}}\sum_{l=1}^{i}\sum_{s=1}^{2^m-1}|\Delta_{ls}^{nm}f(x,y)|$$
$$\leqslant c(\alpha)\left(\frac{1}{2^m-1}\right)^{\frac{\beta}{\alpha+\beta}-\beta}\sum_{i=1}^{2^n-2}\frac{1}{i^{1+\frac{\alpha}{\alpha+\beta}-\alpha}}\sup_{y}\left(\sum_{l=1}^{2^n-1}|\Delta_{l}^{n}f(x,y)_1|\right)^{\frac{\alpha}{\alpha+\beta}}$$
$$\times\sup_{x}\left(\sum_{s=1}^{2^m-1}|\Delta_{s}^{m}f(x,y)_2|\right)^{\frac{\beta}{\alpha+\beta}}$$
$$\leqslant c(\alpha)\left(\frac{1}{2^m-1}\right)^{\frac{\beta}{p(\alpha+\beta)}-\beta}\sum_{i=1}^{2^n-2}\frac{1}{i^{1+\frac{\alpha}{p(\alpha+\beta)}-\alpha}}$$

$$\times \sup_y \left(\sum_{l=1}^{2^n-1} |\Delta_l^n f(x,y)_1|^p \right)^{\frac{\alpha}{p(\alpha+\beta)}}$$

$$\times \left(\sup_x \sum_{s=1}^{2^m-1} |\Delta_s^m f(x,y)_2|^p \right)^{\frac{\beta}{p(\alpha+\beta)}}$$

$$\leqslant c(\alpha) \left(\frac{1}{2^m-1} \right)^{\frac{\beta}{p(\alpha+\beta)}-\beta} \sum_{i=1}^{\infty} \frac{1}{i^{1+\frac{\alpha}{p(\alpha+\beta)}-\alpha}}$$

$$\leqslant c(\alpha) \left(\frac{1}{2^m-1} \right)^{\frac{\beta}{p(\alpha+\beta)}-\beta} \to 0 \quad \text{as } n,m \to \infty. \tag{12}$$

Analogously, we obtain

$$\mathrm{II} \to 0 \quad \text{as } n,m \to \infty. \tag{13}$$

From (10), we write

$$\mathrm{I} \leqslant c(\alpha,\beta) \sum_{i=1}^{2^n-2} \sum_{j=1}^{2^m-2} \frac{1}{i^{2-\alpha}} \frac{1}{j^{2-\beta}} \sum_{l=1}^{i} \sum_{s=1}^{j} |\Delta_{ls}^{nm} f(x,y)|$$

$$\leqslant c(\alpha,\beta) \sum_{i=1}^{2^n-2} \frac{1}{i^{1+\frac{\alpha}{\alpha+\beta}-\alpha}} \sup_y \left(\sum_{l=1}^{i} |\Delta_l^n f(x,y)_1| \right)^{\frac{\alpha}{\alpha+\beta}}$$

$$\times \sum_{j=1}^{2^m-2} \frac{1}{j^{1+\frac{\beta}{\alpha+\beta}-\beta}} \sup_x \left(\sum_{s=1}^{j} |\Delta_s^m f(x,y)_2| \right)^{\frac{\beta}{\alpha+\beta}}$$

$$\leqslant c(\alpha,\beta) \sum_{i=1}^{2^n-2} \frac{1}{i^{1+\frac{\alpha}{p(\alpha+\beta)}-\alpha}} \sup_y \left(\sum_{l=1}^{i} |\Delta_l^n f(x,y)_1|^p \right)^{\frac{\alpha}{p(\alpha+\beta)}}$$

$$\times \sum_{j=1}^{2^m-2} \frac{1}{j^{1+\frac{\beta}{p(\alpha+\beta)}-\beta}} \sup_x \left(\sum_{s=1}^{j} |\Delta_s^m f(x,y)_2|^p \right)^{\frac{\beta}{p(\alpha+\beta)}}. \tag{14}$$

Since

$$\sum_{i=1}^{2^n-2} \frac{1}{i^{1+\frac{\alpha}{p(\alpha+\beta)}-\alpha}} \sup_y \left(\sum_{l=1}^{i} |\Delta_l^n f(x,y)_1|^p \right)^{\frac{\alpha}{p(\alpha+\beta)}}$$

$$= \sum_{i=1}^{i(n)} \frac{1}{i^{1+\frac{\alpha}{p(\alpha+\beta)}-\alpha}} \sup_y \left(\sum_{l=1}^{i} |\Delta_l^n f(x,y)_1|^p \right)^{\frac{\alpha}{p(\alpha+\beta)}}$$

$$+\sum_{i=i(n)+1}^{2^n-2}\frac{1}{i^{1+\frac{\alpha}{p(\alpha+\beta)}-\alpha}}\sup_y\left(\sum_{l=1}^{i}|\Delta_l^n f(x,y)_1|^p\right)^{\frac{\alpha}{p(\alpha+\beta)}}$$
$$\leqslant c(\alpha,\beta,p)\left\{\left[\overset{\bullet}{\omega}_1\left(f,\frac{1}{2^n}\right)\right]^{\frac{\alpha}{\alpha+\beta}}i(n)^\alpha+\left(\frac{1}{i(n)}\right)^{\frac{\alpha}{p(\alpha+\beta)}-\alpha}\right\}=o(1)\quad\text{as } n\to\infty,$$

where

$$\min_{1\leqslant m<2^n}\left\{\left[\overset{\bullet}{\omega}_1\left(f,\frac{1}{2^n}\right)\right]^{\frac{\alpha}{\alpha+\beta}}m^\alpha+\left(\frac{1}{m}\right)^{\frac{\alpha}{p(\alpha+\beta)}-\alpha}\right\}$$
$$=\left\{\left[\overset{\bullet}{\omega}_1\left(f,\frac{1}{2^n}\right)\right]^{\frac{\alpha}{\alpha+\beta}}(i(n)^\alpha)+\left(\frac{1}{i(n)}\right)^{\frac{\alpha}{p(\alpha+\beta)}-\alpha}\right\},$$

from (14) we get

$$\mathrm{I}\leqslant c(\alpha,\beta,p)\left\{\left[\overset{\bullet}{\omega}_1\left(f,\frac{1}{2^n}\right)\right]^{\frac{\alpha}{\alpha+\beta}}(i(n))^\alpha+\left(\frac{1}{i(n)}\right)^{\frac{\alpha}{p(\alpha+\beta)}-\alpha}\right\}$$
$$\times\sum_{j=1}^{\infty}\frac{1}{j^{1+\frac{\beta}{p(\alpha+\beta)}-\beta}}\to 0\quad\text{as } n,m\to\infty. \tag{15}$$

Combining (7), (11)–(13) and (15) yields

$$W_{nm}(f;x,y)\to 0 \tag{16}$$

uniformly with respect to x,y as $m,n\to\infty$.

By (5), (6), (16) and from the Lemma 3 the proof of Theorem 1 is complete. □

Proof of Theorem 2. We choose a monotone increasing sequence of positive integers $\{l_k\colon k\geqslant 0\}$ such that $l_0>s$ (where s is the same as in Lemma 1) and

$$l_k>2l_{k-1}, k\geqslant 1, \tag{17}$$

$$\frac{1}{2^{l_k(2-\alpha-\beta)}}\sum_{i=1}^{k-1}2^{2l_i}\left(\frac{2^{2l_{i-1}}}{2^{l_i}}\right)^{\frac{\alpha+\beta}{p}}<\frac{1}{k}. \tag{18}$$

Consider the function φ_k defined by

$$\varphi_k(x)=\begin{cases}2^{l_k+2}x-2j, x\in[2j2^{-l_k-2},(2j+1)/2^{-l_k-2})\\ -(2^{l_k+2}x-2j-2), x\in[(2j+1)2^{-l_k-2},(2j+2)/2^{-l_k-2})\\ \text{for } j=1,\dots,r(l_k,p)-1,\\ 0 \text{ otherwise},\end{cases}$$

$$\varphi_k(x+l)=\varphi_k(x),\quad l=\pm 1,\pm 2,\dots,$$

where

$$r(l_k,p)=\max\left\{j:\frac{j+1}{2^{l_k/p}}\leqslant\frac{1}{2}\frac{1}{2^{l_{k-1}/p}}\right\}.$$

Let

$$f_0(x,y)=\sum_{n=1}^{\infty}\left(\frac{2^{l_{n-1}}}{2^{l_n}}\right)^{\frac{\alpha+\beta}{p}}\varphi_n^{\alpha}(x)\varphi_n^{\beta}(y),$$

where

$$\varphi_n^{\alpha}(x)=\varphi_n(x)\,\mathrm{sign}\,K_{2^{l_n}}^{-\alpha}(x).$$

It is evident that f_0 is continuous on I^2 and 1-periodic with respect to each variable. Since $\alpha+\beta\geqslant 1/p$, from the construction of the function f_0 we obtain that $f_0\in PBV_p(I^2)$.

We show that the $(C;-\alpha,-\beta)$-means of the double Walsh–Fourier series of the function f_0 diverge over cubes for $(x,y)=(0,0)$.

Indeed,

$$\begin{aligned}
&\sigma_{2^{l_k},2^{l_k}}^{-\alpha,-\beta}(f_0;0,0)-f_0(0,0)\\
&\quad=\int_0^1\int_0^1 f_0(x,y)K_{2^{l_k}}^{-\alpha}(x)K_{2^{l_k}}^{-\beta}(y)\,dx\,dy\\
&\quad=\left(\int_0^{2^{-l_k-1}}\int_0^{2^{-l_k-1}}+\int_{2^{-l_k-1}}^{2^{-l_{k-1}-1}}\int_{2^{-l_k-1}}^{2^{-l_{k-1}-1}}+\int_{2^{-l_{k-1}-1}}^{1}\int_{2^{-l_{k-1}-1}}^{1}\right)\\
&\qquad\times(f_0(x,y)K_{2^{l_k}}^{-\alpha}(x)K_{2^{l_k}}^{-\beta}(y)\,dx\,dy)=\mathrm{I}+\mathrm{II}+\mathrm{III}.
\end{aligned}\tag{19}$$

Since $|K_n^{-\alpha}(x)|=O(n)$, for I we obtain

$$\begin{aligned}
\mathrm{I}&\leqslant\int_0^{2^{-l_k-1}}\int_0^{2^{-l_k-1}}|f_0(x,y)|\,|K_{2^{l_k}}^{-\alpha}(x)|\,|K_{2^{l_k}}^{-\beta}(y)|\,dx\,dy\\
&\leqslant c(\alpha,\beta)\max_{x,y\in[0,2^{-l_k-1}]^2}|f_0(x,y)|=o(1)\quad\text{as } k\to\infty.
\end{aligned}\tag{20}$$

Since

$$\|\mathrm{III}\|_C\leqslant\sum_{i=1}^{k-1}\left(\frac{2^{l_{i-1}}}{2^{l_i}}\right)^{\frac{\alpha+\beta}{p}}\|\varphi_i^{\alpha}-\sigma_{2^{l_k}}^{-\alpha}(\varphi_i^{\alpha})\|_C\|\varphi_i^{\beta}-\sigma_{2^{l_k}}^{-\beta}(\varphi_i^{\beta})\|_C$$

and

$$\omega\left(\varphi_i^\alpha, \frac{1}{2^{l_k}}\right) = O\left(\frac{2^{l_i}}{2^{l_k}}\right),$$

$$\omega\left(\varphi_i^\beta, \frac{1}{2^{l_k}}\right) = O\left(\frac{2^{l_i}}{2^{l_k}}\right),$$

from Lemma 2 and by (18) we obtain

$$\begin{aligned}||\mathrm{III}||_C &\leqslant c(\alpha,\beta)\sum_{i=1}^{k-1}\left(\frac{2^{l_{i-1}}}{2^{l_i}}\right)^{\frac{\alpha+\beta}{p}}\omega\left(\varphi_i^\alpha,\frac{1}{2^{l_k}}\right)\omega\left(\varphi_i^\beta,\frac{1}{2^{l_k}}\right)2^{l_k(\alpha+\beta)}\\ &\leqslant c(\alpha,\beta)\sum_{i=1}^{k-1}\left(\frac{2^{l_{i-1}}}{2^{l_i}}\right)^{\frac{\alpha+\beta}{p}}\frac{2^{l_k(\alpha+\beta)}}{2^{2l_k}}2^{2l_i}\\ &= c(\alpha,\beta)\frac{1}{2^{l_k(2-\alpha-\beta)}}\sum_{i=1}^{k-1}\left(\frac{2^{l_{i-1}}}{2^{l_i}}\right)^{\frac{\alpha+\beta}{p}}2^{2l_i}\leqslant\frac{c(\alpha,\beta)}{k}=o(1)\quad\text{as } k\to\infty. \qquad (21)\end{aligned}$$

From the construction of the function, we obtain

$$\begin{aligned}\mathrm{II} &= \left(\frac{1}{2^{l_k-l_{k-1}}}\right)^{\frac{\alpha+\beta}{p}}\int_{2^{-l_k-1}}^{r(l_k,p)2^{-l_k-1}}\varphi_k^\alpha(x)K_{2^{l_k}}^{-\alpha}(x)\,dx\\ &\quad\times\int_{2^{-l_k-1}}^{r(l_k,p)2^{-l_k-1}}\varphi_k^\beta(y)K_{2^{l_k}}^{-\beta}(y)\,dy\\ &= \left(\frac{1}{2^{l_k-l_{k-1}}}\right)^{\frac{\alpha+\beta}{p}}\int_{2^{-l_k-1}}^{r(l_k,p)2^{-l_k-1}}\varphi_k(x)|K_{2^{l_k}}^{-\alpha}(x)|\,dx\\ &\quad\times\int_{2^{-l_k-1}}^{r(l_k,p)2^{-l_k-1}}\varphi_k(y)|K_{2^{l_k}}^{-\beta}(y)|\,dy. \qquad (22)\end{aligned}$$

By Lemma 1 we write

$$\begin{aligned}&\int_{2^{-l_k-1}}^{r(l_k,p)2^{-l_k-1}}\varphi_k(x)|K_{2^{l_k}}^{-\alpha}(x)|\,dx\\ &\quad\geqslant c\sum_{d=1}^{[\log_2 r(l_k,p)]}\int_{\frac{2^{d-1}}{2^{l_k+1}}}^{\frac{2^d}{2^{l_k+1}}}|K_{2^{l_k}}^{-\alpha}(x)|\,dx\\ &\quad\geqslant c(\alpha)\sum_{d=1}^{[\log_2 r(l_k,p)]}2^{\alpha d}\geqslant c(\alpha)(r(l_k,p))^\alpha\geqslant c(\alpha)\left(\frac{2^{l_k}}{2^{l_{k-1}}}\right)^{\frac{\alpha}{p}}. \qquad (23)\end{aligned}$$

Analogously we obtain

$$\int_{2^{-l_k-1}}^{r(l_k,p)2^{-l_k-1}} \varphi_k(y)|K_{2^{l_k}}^{-\beta}(y)|\,dy \geqslant c(\beta)\left(\frac{2^{l_k}}{2^{l_{k-1}}}\right)^{\frac{\beta}{p}}. \tag{24}$$

After substituting (23) and (24) in (22) we have

$$|\mathrm{II}| \geqslant c(\alpha,\beta) > 0. \tag{25}$$

Owing to (19)–(21) and (25), we arrive at

$$\overline{\lim_{k\to\infty}} |\sigma_{2^{l_k},2^{l_k}}^{-\alpha,-\beta}(f_0,0,0) - f_0(0,0)| \geqslant c(\alpha,\beta) > 0.$$

The proof of Theorem 2 is complete. □

Observe that the result of this paper can be proved in the same way for dimension more than 2.

References

[1] B.I. Golubov, A.V. Efimov, V.A. Skvortsov, Series and Transformations of Walsh, Nauka, Moscov, 1987 (Russian), English transl.: Kluwer Academic publishers, Dordrecht, 1991.
[2] U. Goginava, On the uniform convergence of multiple trigonometric Fourier series, East J. Approx. 5 (1999) 253–266.
[3] U. Goginava, On the convergence and summability of N-dimensional Fourier series with respect to the Walsh–Paley systems in the spaces $L^p([0,1]^N), p\in[1,\infty]$, Georgian Math. J. 7 (2000) 53–72.
[4] U. Goginava, On the uniform summability of multiple Walsh–Fourier series, Anal. Math. 26 (2000) 209–226.
[5] U. Goginava, On the approximation properties of Cesàro means of negative order of Walsh–Fourier series, J. Approx. Theory 115 (2002) 9–20.
[6] G.H. Hardy, On double Fourier series, Quart. J. Math. 37 (1906) 53–79.
[7] C. Jordan, Sur le series de Fourier, C. R. Acad. Sci. Paris 92 (1881) 228–230.
[8] F. Moricz, On the uniform convergence and L^1-convergence of double Walsh–Fourier series, Studies Math. 102 (1992) 225–237.
[9] A. Paley, A remarkable series of orthogonal functions, Proc. London Math. Soc. 34 (1932) 241–279.
[10] F. Schipp, W.R. Wade, P. Simon, J. Pàl, Walsh Series, Introduction to Dyadic Harmonic Analysis, Adam Hilger, Briston, New York, 1990.
[11] V.I. Tevzadze, Uniform convergence of Cesàro means of negative order of Fourier Walsh series, Soobshch. Acad. Nauk Gruzin. SSR 102 (1981) 33–36 (in Russian).